EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS
Earthquake Engng. Struct. Dyn. **27**, 415–421 (1998)

ESTIMATED MASS AND STIFFNESS MATRICES OF SHEAR BUILDING FROM MODAL TEST DATA

PING YUAN,[1]* ZHIFENG WU[2] AND XINGRUI MA[2]

[1]*Real Estate Department, Room 621, Tonge Group Co. Ltd., PO Box 34-108, ShenZhen GuangDong 518034, Peoples' Republic of China*
[2]*Department of Civil Engineering, Harbin Institute of Technology, Harbin, 150001, Peoples' Republic of China*

SUMMARY

A method that estimates mass and stiffness matrices of shear building from modal test data is presented in this paper. The method depends on only measurable points that are less in number than the total structural degrees of freedom, and on the first two orders of structural mode measured. So it is applicable to most of the general test. Based on this method modal data of unmeasurable points are estimated, then global mass and stiffness matrices of structure are obtained by using the first two orders of modal data. Taking advantage of iteration the optimum global mass and stiffness matrices are gained. Finally, an example is studied in this paper. Its result shows that this method is reliable. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: modal analysis; shear structures stiffness; multidegree-of-freedom systems analysis

1. INTRODUCTION

Determination of the dynamic characteristics including natural frequencies and mode shapes of structure from test data is developed in recent years. Naturally, only a finite number of points on the structure can be collected, and these points are generally a small subset of the total degrees of freedom (DOF) in a finite element model of the structure. In fact, the number of measurement points may also be less than the total number of structural DOF. Therefore, if l sensors are used to identify modal data for n DOF, where $n > l$, there is not a unique model of the classical mass/stiffness form.

In recent years several researchers have focused on methods to reduce structural DOF, so that mass and stiffness can be estimated by limited sensors and frequency data obtained from modal testing.[1–3]

The primary goal of the present paper is to investigate a direct solution to an inverse vibration problem. In this problem, the number of sensors l is less than the structural DOF n. We will show that mass and stiffness matrices of dimension n, referring to the total DOF, can be evaluated from the modal data of l points.

In this paper, the unmeasured modal data are evaluated by the measured modal data. Finally, we obtain the mass and stiffness matrices by using the first two-order modes.

2. EVALUATION OF UNMEASURED MODAL DATA

Recent work in the area of structural identification has included the determination of mass and stiffness matrices directly from continuous-time system realizations.[4] This approach requires that the number of independent sensors be equal to the number of identified modes. A more practical approach is to enrich the

* Correspondence to: Ping Yuan, Real Estate Department, Room 621, Tonge Group Co. Ltd., PO Box 34-108, ShenZhen GuangDong 518034, Peoples' Republic of China

CCC 0098–8847/98/050415–07$17·50

Received 28 October 1996
Revised 7 April 1997

computer mass and stiffness matrices with the complete set of measured modes, independent of the number of sensors. We begin by developing the concept of evaluating the unmeasured modal data from the measured modal data.

If the structure is regarded as a shear building, we can assume that (1) the total mass of the structure is concentrated at the levels of the floors; (2) the floors are infinitely rigid as compared to the columns; and (3) the deformation of the structure is independent of the axial forces in the columns. These assumptions transform the problem from a structure with an infinite number of degrees of freedom into a structure with as many DOF as numbers of floor levels. Clearly, the stiffness matrix of a shear building is a tridiagonal matrix, and the mass matrix is a diagonal matrix. The mass and stiffness of a shear building can be written as

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & \cdots & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & & \vdots \\ \vdots & & \ddots & & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & \cdots & \cdots & -k_n & k_n \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & & \cdots & & 0 \\ & m_2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & m_{n-1} & \\ 0 & & \cdots & & m_n \end{bmatrix}$$

Tall building always have standard floor levels with the same mass and stiffness. Assumed that the first standard floor level is i, $i \neq 1$, and the last standard floor level is j, $j \leqslant n$, where n is the total floor levels of building. In a word there are $j - i = d$ standard floor levels, and the first floor level is not standard level. If sensors are placed at $i - 1$ and j floor levels, the modal data obtained can be used to evaluate modal data of the unmeasured points.

MDOF is an arbitrary undamped system, the differential equation of the system is given by

$$M\ddot{X} + KX = F$$

where M is the structural mass matrix, K the structural stiffness matrix, X the vector of generalized modal deflection, $\ddot{X}$ the vector of generalized modal accelerations, and F the vector of external forces. Consider the corresponding characteristics equation:

$$(K - \lambda_l M)\phi_l = 0 \tag{1}$$

where λ_l is the lth eigenvalue, and ϕ_l the lth corresponding mode shape or eigenvector. Equation (1) can also be written as

$$\left(\begin{bmatrix} k_{aa} & k_{ab} & 0 \\ k_{ba} & k_{bb} & k_{bc} \\ 0 & k_{cb} & k_{cc} \end{bmatrix} - \lambda_l \begin{bmatrix} m_{aa} & 0 & 0 \\ 0 & m_{bb} & 0 \\ 0 & 0 & m_{cc} \end{bmatrix} \right) \begin{Bmatrix} \phi_a \\ \phi_b \\ \phi_c \end{Bmatrix} = 0 \tag{2}$$

where $a = 1, 2, \ldots, i-1$, $b = i, i+1, \ldots, j-1$, $c = j, j+1, \ldots, n$, mass matrices m_{aa}, m_{bb}, m_{cc} are diagonal matrices, and stiffness matrices can be written as

$$k_{ba} = \begin{bmatrix} 0 & \cdots & -k_i \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = k_{ab}^{\mathrm{T}}$$

and

$$k_{bc} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -k_j & \cdots & 0 \end{bmatrix} = k_{cb}^{\mathrm{T}} \tag{3}$$

Equation (2) is partition of equation (1). The second line of equation (2) can be expressed as

$$k_{ba}\phi_a + k_{bb}\phi_b + k_{bc}\phi_c - \lambda_l m_{bb}\phi_b = 0 \tag{4}$$

Substituting equation (3) into equation (4), we have

$$k_{bb}\phi_b - \lambda_l m_{bb}\phi_b = \begin{Bmatrix} k_1\phi_{i-1} \\ 0 \\ \vdots \\ 0 \\ k_j\phi_j \end{Bmatrix}_{d\times 1} \tag{5}$$

Expanding equation (5), we have

$$\begin{aligned} (k_i + k_{i+1})\phi_i - k_{i+1}\phi_{i+1} - \lambda_l m_i\phi_i &= k_i\phi_{i-1} \\ -k_{i+1}\phi_i + (k_{i+1} + k_{i+2})\phi_{i+1} - k_{i+2}\phi_{i+2} - \lambda_l m_{+1i}\phi_{i+1} &= 0 \\ \vdots \quad & \quad \vdots \\ -k_{j-1}\phi_{j-2} + (k_{j-1} + k_j)\phi_{j-1} - \lambda_l m_{j-1}\phi_{j-1} &= k_j\phi_j \end{aligned} \tag{6}$$

Because of standard levels there are the following relationships:

$$\begin{aligned} k &= k_i = k_{i+1} = \cdots = k_j \\ m &= m_i = m_{i+1} = \cdots = m_j \end{aligned} \tag{7}$$

Substituting equation (7) into equation (6), equation (6) simplifies to

$$A\{\phi_b\} = \left(\begin{bmatrix} 2-\lambda_l\alpha & -1 & \cdots & 0 \\ -1 & 2-\lambda_l\alpha & -1 & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & -1 & 2-\lambda_l\alpha \end{bmatrix} \right) \begin{Bmatrix} \phi_i \\ \phi_{i+1} \\ \vdots \\ \phi_{j-1} \end{Bmatrix} = \begin{Bmatrix} \phi_{i-1} \\ 0 \\ \vdots \\ \phi_j \end{Bmatrix} = D \tag{8}$$

where $\alpha = m/k$, $\{\phi_b\} = \{\phi_i, \phi_{i+1}, \ldots, \phi_{j-1}\}^{\mathrm{T}}$, $D = \{\phi_{i-1}, 0, \ldots, \phi_j\}^{\mathrm{T}}$, and A is a $(j-i)\times(j-i)$ matrix, using least-squares approach the unmeasured modal shape $\{\phi_b\}$ can be obtained:

$$\begin{aligned} A\{\phi_b\} &= D \\ A^{\mathrm{T}}A\{\phi_b\} &= A^{\mathrm{T}}D \end{aligned} \tag{9a}$$

$$\{\phi_b\} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}D \tag{9}$$

By means of equation (9) modal data from i to $j-1$ can be evaluated by the $(i-1)$th and the jth level modal data.

3. THE EVALUATION OF MASS AND STIFFNESS MATRICES

Expanding equation (2) we have

$$\begin{gathered}
(k_1 + k_2)\phi_{l1} - k_2\phi_{l2} - \lambda_l m_1 \phi_{l1} = 0 \\
- k_2\phi_{l1} + (k_2 + k_3)\phi_{l2} - k_3\phi_{l3} - \lambda_l m_2\phi_{l2} = 0 \\
\vdots \quad \vdots \\
- k_{n-1}\phi_{\mathrm{ln}-2} + (k_{n-1} + k_n)\phi_{\mathrm{ln}-1} - k_n\phi_{\mathrm{ln}} - \lambda_l m_{n-1}\phi_{\mathrm{ln}-1} = 0 \\
- k_n\phi_{\mathrm{ln}-1} + k_n\phi_{\mathrm{ln}} - \lambda_l m_n\phi_{\mathrm{ln}} = 0
\end{gathered} \tag{10}$$

In order to evaluate mass and stiffness matrices, equation (10) can be written for modes l and r as

$$\begin{bmatrix}
\phi_{l1} & -\lambda_l\phi_{l1} & \phi_{l1} - \phi_{l2} & & & & \cdots & & 0 \\
\phi_{r1} & -\lambda_r\phi_{r1} & \phi_{r1} - \phi_{r2} & & & & & & \\
 & & \phi_{l2} - \phi_{l1} & \lambda_l\phi_{l2} & \phi_{l2} - \phi_{l3} & & & & \\
 & & \phi_{r2} - \phi_{r1} & \lambda_r\phi_{r2} & \phi_{r2} - \phi_{r3} & & & & \\
\vdots & & \cdots & & \ddots & & \cdots & & \vdots \\
 & & & & & \phi_{\mathrm{ln}-1} - \phi_{\mathrm{ln}-2} & \lambda_l\phi_{\mathrm{ln}-1} & \phi_{\mathrm{ln}-1} - \phi_{\mathrm{ln}} & \\
 & & & & & \phi_{m-1} - \phi_{m-2} & \lambda_r\phi_{m-1} & \phi_{m-1} - \phi_{m} & \\
 & & & & & & & \phi_{\mathrm{ln}} - \phi_{\mathrm{ln}-1} & \lambda_l\phi_{\mathrm{ln}} \\
 & & & & & & & \phi_{m} - \phi_{m-1} & \lambda_r\phi_{m}
\end{bmatrix}
\times \begin{Bmatrix} k_1 \\ m_1 \\ k_2 \\ m_2 \\ \vdots \\ \vdots \\ \vdots \\ k_n \\ m_n \end{Bmatrix} = 0 \tag{11}$$

where r is the rth mode. This equation can be simply written as

$$B\{b\} = 0 \tag{12}$$

where B is an $n \times n$ matrix, $\{b\} = \{k_1, m_1, \dots, k_n, m_n\}^{\mathrm{T}}$. In order to solve this equation the number of equations should be equal to the number of unknowns. Because the number of equations are $2 \times n$, we use the two-order modes. Clearly, matrix B is an singular matrix. Let $m_n = 1$, then equation (12) can be expressed as

$$B'\{b'\} = \begin{Bmatrix} 0 \\ \vdots \\ \lambda_l\phi_{\mathrm{ln}} \\ \lambda_r\phi_{rn} \end{Bmatrix} \tag{13}$$

where B' is the matrix in which the last two rows and the last column of matrix B are eliminated. $\{b'\}$ is the vector in which the last member of vector $\{b\}$ is eliminated. Using the least-squares approach $\{b'\}$ can be obtained:

$$B'^{\mathrm{T}}B'\{b'\} = B'^{\mathrm{T}}\begin{Bmatrix} 0 \\ \vdots \\ \lambda_l \phi_{1n} \\ \lambda_r \phi_{rn} \end{Bmatrix} \tag{14a}$$

$$\{b'\} = (B'^{\mathrm{T}} \cdot B')^{-1} B'^{\mathrm{T}} \begin{Bmatrix} 0 \\ \vdots \\ \lambda_l \phi_{1n} \\ \lambda_r \phi_{rn} \end{Bmatrix} \tag{14}$$

The mass and stiffness matrices obtained by equation (14) are relative values.

The following procedure has been developed for evaluating the global mass and stiffness matrices of the shear building.

Step 1. Choose the initial parameter α_0, α is the ratio between the mass and the stiffness of standard floor levels. We define the initial parameter $\alpha_0 = 1/\lambda_1$.

Step 2: Evaluate the mode shapes of unmeasured DOF.

In this step, equation (9) is used to estimate the two-order mode shapes $\{\phi_b\}_l$ and $\{\phi_b\}_r$, where $\{\phi_b\}_l$ and $\{\phi_b\}_r$ are the lth and the rth corresponding mode shapes.

Step 3: Use equation (11) to evaluate mass and stiffness matrices. Substituting $\{\phi_b\}_l$, $\{\phi_b\}_r$, and corresponding eigenvalues into equation (11), the mass and stiffness coefficients for all DOF can be obtained.

Step 4: Use orthogonality of mass matrix, the following equation is given:

$$f(\alpha_0) = \{\phi\}_l^{\mathrm{T}} [M] \{\phi\}_r$$

Step 5: Assume $\alpha_1 = \alpha_0 + \Delta\alpha$, then return to step 2, $f(\alpha_1)$ is obtained from step 4. If the difference between $f(\alpha_0)$ and $f(\alpha_1)$ is greater than a specified tolerance, return to step 2 with $\alpha_2 = \alpha_1 + \Delta\alpha$, $f(\alpha_0) = f(\alpha_1)$. The procedure cycles between steps 2–5 until it converges. This specified tolerance can be written as

$$f(\alpha_k) - f(\alpha_{k-1}) \leqslant \mathrm{Esp}$$

If the mass and stiffness distributions are assigned to the standard floor levels, we can use this method too. Equation (7) can be written as

$$\begin{aligned} \{k_i, k_{i+1}, \ldots, k_j\}^{\mathrm{T}} &= k\{p_i, p_{i+1}, \ldots, p_j\}^{\mathrm{T}} = k\{p\} \\ \{m_i, m_{i+1}, \ldots, m_j\}^{\mathrm{T}} &= m\{q_i, q_{i+1}, \ldots, q_j\}^{\mathrm{T}} = m\{q\} \end{aligned} \tag{15}$$

where $\{p\}$, $\{q\}$ are distributions of structural mass and stiffness. Using these equations, equation (8) can be written as

$$A\{\phi_b\} = \left(\begin{bmatrix} (p_i + p_{i+1}) - \lambda_i q_i \alpha & -p_{i+1} & \cdots & 0 \\ -p_{i+1} & (p_{i+1} + p_{i+2}) - \lambda_l q_{i+1}\alpha & -p_{i+2} & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & -p_{j-1} & (p_{j-1} + p_j) - \lambda_l q_{j-1}\alpha \end{bmatrix} \right)$$

$$\times \begin{Bmatrix} \phi_i \\ \phi_{i+1} \\ \vdots \\ \phi_{j-1} \end{Bmatrix} = \begin{Bmatrix} p_i \phi_{i-1} \\ 0 \\ \vdots \\ p_j \phi_j \end{Bmatrix} = D \tag{16}$$

where A is $(j-i)\times(j-i)$ matrix, and equations (9)–(14) are the same.

4. EXAMPLE

The example is a shear building with four DOF, each floor level of which is associated with one DOF. Sensors are placed at the first floor level and the fourth floor level. Assuming standard floor levels are the 2–4, and the first and the fifth floor levels are evaluated. The measured eigenvalues and eigenvectors are listed in Table I.

Choosing the initial parameter $\alpha_0 = 0$. After the 12th iteration, the mass and stiffness matrices converge. The procedure of evaluation is shown in Figures 1 and 2. The evaluated values and real values are listed in Table II. Clearly, there is a constant ratio 2×18000 between the evaluated and real values.

Table I. Original data

	First level		Fourth level	
	Frequency	Eigenvector	Frequency	Eigenvector
First order	1·3221	0·6337	1·3221	− 0·5820
Second order	2·7802	0·3698	2·7802	0·5370

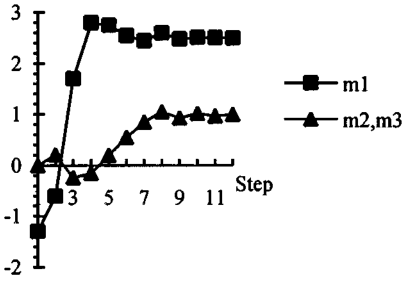

Figure 1. Evaluation of mass

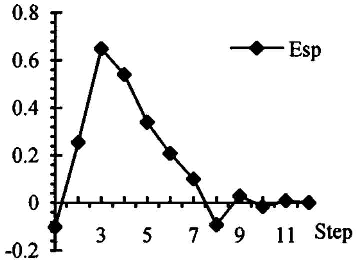

Figure 2. Esp of iteration

Table II. Values of mass and stiffness

	k_1(N/m)	k_2	k_3	k_4	m_1(kg)	m_2	m_3	m_4
Real value	18 000 × 6	18 000 × 3	18 000 × 3	18 000 × 3	18 000 × 5	18 000 × 2	18 000 × 2	18 000 × 2
Evaluation	2·9948	1·4994	1·4994	1·4994	2·4991	0·99896	0·99896	1·0

5. CONCLUSIONS

A new method that evaluates mass and stiffness matrices of a shear building from modal test data is presented in this paper. The approach is based on the number of measurement points being less than the total number of degrees of freedom in the structure, and its relationship to the Kuhar reduction technique for finite element analysis. In this method only the first two-order modes are used to solve the inverse vibration problem. By means of this method it is possible to use modal test data to evaluate the global mass and stiffness matrices. An example is presented to show the method to be reliable.

REFERENCES

1. A. M. Kabe, 'Stiffness matrix adjustment using mode data', *AIAA* **23** (9), 1431–1436 (1985).
2. C. Farhat, and F. Hemez, 'Updating finite element dynamic models using an element-by-element sensitivity methodology', *AIAA* **31** (9), 1702–1711 (1993).
3. C. D. Yang, and F. B. Yeh, Identification, reduction, and retinement of model parameters by the eigensystem realization algorithm', *J. Guidance Control Dyn.* **13** (6), 1051–1059 (1990).
4. M. Baruch, 'Optimization procedure to correct stiffness and flexibility matrices using vibration tests, *AIAA* **16** (11), 1209–1210 (1978).
5. K. G. Topole and N. Stubbs, 'Non-destructive damage evaluation of a structure from limiten modal parameters', *Earthquake Engng. Struct. Dyn.* **24** (11), 1427–1436 (1995).

